NONLINEAR BEHAVIOR, ANALYSIS, AND DESIGN

Most civil engineering structures behave in a linear elastic fashion under service loads. Exceptions are slender structures such as some suspension systems, arches, and tall buildings, and structures subject to early localized yielding or cracking. But prior to reaching their limit of resistance, almost all structures would exhibit significant nonlinear response. Therefore, if linear elastic analysis is the highest level available, the design engineer must find another way to account for the effects that the analysis is incapable of simulating. The answer may lie in the following:

a) individual judgment
b) code formulas that accept the results of a linear elastic or simpler analysis and make allowance for nonlinearity in some empirical or semi-empirical way
c) supplementary theoretical or experimental analysis

In nonlinear analysis an attempt is made to improve the analytical simulation of the behavior of a structure in some respect. The fundamental aim is to improve the quality of design by providing the engineer with a more reliable prediction of the performance of a system that is under design or investigation.

SOURCES OF NONLINEARITY

In linear elastic analysis, the material is assumed to be unyielding and its properties invariable and the equations of equilibrium are formulated on the geometry of the unloaded structure or, in the case of self-strained structures, on an initial reference configuration. Subsequent deformations are assumed to be so small as to be insignificant in their effect on the equilibrium and mode of response of the system. One consequence of this was our ability to treat axial force, bending moments, and torques as uncoupled actions in developing the stiffness equations for frame finite elements of bisymmetrical sections.

Nonlinear analysis offers several options:

- Only geometric nonlinearity may be considered. That is, treat the material as elastic but include the effects of deformations and finite displacements in formulation of equations of equilibrium.
- Only material nonlinearity may be considered. That is, the effect of changes of material properties under load.
- Both the effects of material and geometric nonlinearity may be considered.

Geometrical effects:

1) Initial imperfections such as member camber and out-of-plumb erection of a frame.
2) P-Δ effect, a destabilizing moment equal to a gravity load times the horizontal displacement it undergoes as a result of the lateral displacement of the supporting structure.
3) P-δ effect, the influence of axial force on the flexural stiffness of an individual member.

**Material effects:**
1) Plastic deformation of steel structures
2) Cracking and creep of reinforced concrete structures
3) Inelastic interaction of axial force, bending, shear, and torsion

**Combined effects:**
1) Plastic deformation plus P-δ and P-Δ effects.
2) Connection deformations
3) Panel zone deformations.
4) Contributions of infilling and secondary systems to strength and stiffness.

**Engineering Applications of Nonlinear Structural Analysis:**

<table>
<thead>
<tr>
<th>Application</th>
<th>Explanation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Strength analysis</td>
<td>How much load can the structure support before global failure occurs?</td>
</tr>
<tr>
<td>Deflection analysis</td>
<td>When deflection control is of primary importance</td>
</tr>
<tr>
<td>Stability analysis</td>
<td>Finding critical points (limit points or bifurcation points) closest to operational range</td>
</tr>
<tr>
<td>Service configuration analysis</td>
<td>Finding the “operational” equilibrium form of certain slender structures when the fabrication and service configurations are quite different (e.g. cables, inflatable structures, helicoids)</td>
</tr>
<tr>
<td>Reserve strength analysis</td>
<td>Finding the load carrying capacity beyond critical points to assess safety under abnormal conditions.</td>
</tr>
<tr>
<td>Progressive failure analysis</td>
<td>A variant of stability and strength analysis in which progressive deterioration (e.g. cracking) is considered.</td>
</tr>
<tr>
<td>Envelope analysis</td>
<td>A combination of previous analyses in which multiple parameters are varied and the strength information thus obtained is condensed into failure envelopes.</td>
</tr>
</tbody>
</table>

**TERMINOLOGY**

**Equilibrium Path:** The concept of *equilibrium path* plays a central role in explaining the mysteries of nonlinear structural analysis. This concept lends itself to graphical representation in the form of *response diagrams*. The most widely used form of these pictures is the *load-deflection* response diagram. Through this representation many key concepts can be illustrated and interpreted in physical, mathematical or computational terms.
**Load-deflection response:** The gross or overall static behavior of many structures can be characterized by a load-deflection or force-displacement response. The response is usually drawn in two dimensions as a $x$-$y$ plot as shown below. In this figure a “representative” force quantity is plotted against a “representative” displacement quantity. If the response plot is nonlinear, the structure behavior is nonlinear.

A smooth curve shown in a load-deflection diagram is called a *path*. Each point in the path represents a possible *configuration* or *state* of the structure. If the path represents configurations of static equilibrium it is called an *equilibrium path*. Each point in an equilibrium path is called an *equilibrium point*. An equilibrium point is the graphical representation of an *equilibrium state* or *equilibrium configuration*.

The origin of the response plot (zero load, zero deflection) is called the *reference state* because it is the configuration from which loads and deflections are measured. However, the reference state may be in fact chosen rather arbitrarily, and this freedom is exploited in some nonlinear formulations and solution methods, as we shall see later.

For problems involving *perfect* structures the reference state is unstressed and undeformed, and is also an equilibrium state. This means that an equilibrium path passes through the reference state.

The path that crosses the reference state is called the *fundamental equilibrium path* or *fundamental path* for short. (Many authors also call this a *primary path*. ) The fundamental path extends from the reference state up to special states called *critical points*. Any path that is not a fundamental path but connects with it at a critical point is called a *secondary path*.

**Special Equilibrium Points:**
Certain points of an equilibrium path have special significance in the applications and thus receive special names.

**Critical Points:** 1) *Limit points*, at which the tangent to the equilibrium path is horizontal, i.e. parallel to the deflection axis; 2) *Bifurcation points*, at which two or more equilibrium paths cross. At critical points the relation between the given characteristic load and the associated deflection is not unique. Physically, the structure becomes *uncontrollable*.
or marginally controllable there. This property endows such points with engineering significance.

**Turning points:** Points at which the tangent to the equilibrium path is vertical, *i.e.* parallel to the load axis, are called *turning points*. These are not critical points and have less physical significance, but are of interest for some structures. They have some computational significance, however, because they can affect the performance of certain solution methods.

**Failure points:** Points at which a path suddenly stops or “breaks” because of physical failure are called *failure points*. The phenomenon of failure may be *local* or *global* in nature. In the first case (*e.g.*, failure of a noncritical structure component) the structure may regain functional equilibrium after dynamically “jumping” to another equilibrium path. In the latter case the failure is catastrophic or destructive and the structure does not regain functional equilibrium.

**Tangent Stiffness:** The tangent to an equilibrium path may be informally viewed as the limit of the ratio

\[
\frac{\text{Force increment}}{\text{Displacement increment}}
\]

This is by definition a *stiffness* or, more precisely, the *tangent stiffness* associated with the representative force and displacement. The reciprocal ratio is called *flexibility*. The sign of the tangent stiffness is closely associated with the question of *stability* of an equilibrium state. A negative stiffness is necessarily associated with *unstable* equilibrium. A positive stiffness is necessary but not sufficient for stability.

**TYPES OF NONLINEAR RESPONSE**

The response diagrams in below figure illustrate three “monotonic” types of material response: linear, hardening, and softening. Symbols F and L identify failure and limit points, respectively.

A response such as in (a) is characteristic of pure crystals, glassy, and certain high strength composite materials.

A response such as in (b) is typical of cable, netted and pneumatic (inflatable) structures, which may be collectively called *tensile structures*. The stiffening effect comes from geometry “adaptation” to the applied loads. Some flat-plate assemblies also display this behavior initially.
A response such as in (c) is more common for structural materials than the previous two. A linear response is followed by a softening regime that may occur slowly or suddenly. More “softening responses” are given in next figure.

The diagrams above illustrate a “combination of basic responses” that can complicate the response as well as the task of the analyst. Here B and T denote bifurcation and turning points, respectively.

The snap-through response (d) combines softening with hardening following the second limit point. The response branch between the two limit points has a negative stiffness and is therefore unstable. (If the structure is subject to a prescribed constant load, the structure “takes off” dynamically when the first limit point is reached.) A response of this type is typical of slightly curved structures such as shallow arches.

The snap-back response (e) is an exaggerated snap-through, in which the response curve “turns back” in itself with the consequent appearance of turning points. The equilibrium between the two turning points may be stable and consequently physically realizable. This type of response is exhibited by trussed-dome, folded and thin-shell structures in which “moving arch” effects occur following the first limit point; for example cylindrical shells with free edges and supported by end diaphragms.

In all previous diagrams the response was a unique curve. The presence of bifurcation (popularly known as “buckling” by structural engineers) points as in (f) and (g) introduces more features. At such points more than one response path is possible. The structure takes the path that is dynamically preferred (in the sense of having a lower energy) over the others. Bifurcation points may occur in any sufficiently thin structure that experiences compressive stresses. Bifurcation, limit and turning points may occur in many combinations as illustrated in (g). A striking example of such a complicated response is provided by thin cylindrical shells under axial compression.

Rarely, if ever, is it possible to model all sources of nonlinearity and portray the actual behavior of a practical structure in all of its detail. The most common levels of analysis are represented in the next figure by schematic response curves for a statically loaded frame.

The path of increasingly nonlinear response (elastic or inelastic) culminating in instability is probably the most common mode of failure in civil engineering structures.
References:
1) W. McGuire, R.H. Gallagher, R.D. Ziemian; Matrix Structural Analysis
2) C. Felippa; Nonlinear Finite Element Methods (Lecture Notes)